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The M5-brane on $K3 \times T^2$

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Abstract: We discuss the low energy effective theory of an M5-brane wrapped on a smooth holomorphic four-cycle of $K3 \times T^2$, including the special case of T^6. In particular we give the lowest order equations of motion and resolve a puzzle concerning the counting of massless modes that was reported in hep-th/9906094. In order to find agreement with black hole entropy and anomaly inflow arguments we propose that some of the moduli become massive.

Keywords: p-branes, Black Holes in String Theory, M-Theory.

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Contents

1. Introduction

One of the most exciting achievements in string theory is the remarkable success in counting microscopic counting of black hole states, starting with the work of [1]. A particularly elegant example of this is provided by considering an M5-brane wrapped on a complex four-cycle of a Calabi-Yau [2]. This yields a black string in five dimensions which can be further reduced to four dimensions by wrapping the string on S^1 and including momentum along the S^1. A notable feature of this analysis is that, for a generic four-cycle, the M5-brane has a smooth worldvolume and hence the only microscopic information needed is a knowledge of the worldvolume fields and dynamics of a single M5-brane.

There have been several detailed accounts of the M5-brane wrapped on cycles of a generic Calabi-Yau manifold, for example see [3 – 7], and also the M5-brane on $K3$ [8]. The case that we are interested in here concerns an M5-brane whose worldvolume has non-trivial one-cycles which occurs when the Calabi-Yau degenerates to $K3 \times T^2$ or T^6 (see also [7]). This situation was discussed in [9] where several puzzles arose. In particular the number of massless states was not found to be in accordance with $(0,4)$ supersymmetry and the counting of black hole microstates failed (albeit at sub-leading order). To resolve these problems the authors of [9] proposed a novel mechanism whereby some massless modes are charged with respect to the worldvolume gauge fields that arise from reduction of the two-form. The main purpose of this paper is to investigate this proposal. However we find that the correct resolution comes from including additional massless modes which are present when the Calabi-Yau is $K3 \times T^2$ or T^6.

The rest of this paper is organized as follows. In section two we present the lowest order equations of motion for an M5-brane which is wrapped on a smooth cycle P in spacetime. In section three we consider in detail the case where spacetime is of the form $\mathcal{M} = \mathbf{R}^{1,4} \times K3 \times T^2$ and $\mathcal{M} = \mathbf{R}^{1,4} \times T^6$. We provide a careful counting of the normal

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bundle moduli and resolve a puzzle concerning $(0,4)$ supersymmetry that was observed in [9]. In section four we consider four-dimensional black hole states that arise by further compactification on S^1. We find that, using our analysis, the usual counting of left-moving massless modes to determine black hole entropy does not agree with the supergravity calculations or arguments using anomalies. To resolve this discrepancy we propose that $h_{1,0}(P)$ $(4,4)$ multiplets must become massive and hence do not appear in the low energy effective action. This provides an alternative resolution to a second puzzle discussed in [9]. Finally section five contains a brief conclusion.

2. Lowest order equations of motion

Covariant equations of motion of the M5-brane were first derived in [10]. We will not need give the full non-linear form of these equations, however it will be enlightening to give the lowest order equations (in terms of a derivative expansion). We will work in static gauge where the six coordinates x^μ, $\mu,\nu = 0,1,2,\ldots,5$, of the M5-brane worldvolume are identified with the first six coordinates of spacetime. The massless fields consist of 5 scalars X^A, $A,B = 6,7,8,\ldots,10$, a two-form $B_{\mu\nu}$ and a Fermion ψ which satisfies $\Gamma_{012345}\psi = -\psi$. Here we use a full 32-component spinor of $\mathrm{SO}(1,10)$. It will be sufficient to work at the lowest order in the fields. X^A represents the coordinates of the M5-brane in the transverse space and in particular $X^A = 0$ corresponds to the M5-brane wrapped on a calibrated submanifold. We use $M,N = 0,1,2,\ldots,10$ to denote all eleven coordinates. We use an underline to denote tangent space indices. We will use a hat to denote eleven-dimensional quantities, the spacetime is denoted by $\mathcal{M}$ and the M5-brane worldvolume by $\mathcal{W}$.

First recall the case where the worldvolume $\mathcal{W}$ admits a chiral Killing spinor ϵ; $D_\mu\epsilon = 0$, $\Gamma_{012345}\epsilon = \epsilon$. For example if $\mathcal{M} = \mathbf{R}^{1,4}\times K3\times T^2$ and $\mathcal{W} = \mathbf{R}^{1,1}\times K3$. To lowest order in fluctuations, the equations of motion are just that of a free theory on a curved background

$$\begin{aligned} D^2 X^{\underline{A}} &= 0 \\ i\Gamma^\mu D_\mu\psi &= 0 \\ H_{\mu\nu\lambda} &= \frac{1}{3!}\epsilon_{\mu\nu\lambda\rho\sigma\tau}H^{\rho\sigma\tau}, \end{aligned} \tag{2.1}$$

where $H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}$ and $\epsilon_{\mu\nu\lambda\rho\sigma\tau}$ is totally antisymmetric with $\epsilon^{\underline{012345}} = 1$. These equations are invariant under the supersymmetry transformations

$$\begin{aligned} \delta X^{\underline{A}} &= i\bar\epsilon\Gamma^{\underline{A}}\psi \\ \delta B_{\mu\nu} &= i\bar\epsilon\Gamma_{\mu\nu}\psi \\ \delta\psi &= \partial_\mu X^{\underline{A}}\Gamma^\mu\Gamma_{\underline{A}}\epsilon + \frac{1}{2\cdot 3!}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\epsilon. \end{aligned} \tag{2.2}$$

Next we consider the case where the spacetime $\mathcal{M}$ admits a chiral covariantly constant spinor $\hat\epsilon$, $\hat D_M\hat\epsilon = 0$, $\Gamma_{012345}\hat\epsilon = \hat\epsilon$ but where this does not descend to a Killing spinor on $\mathcal{W}$. For example we can take $\mathcal{M} = \mathbf{R}^{1,4}\times K3\times T^2$ but with $\mathcal{W} = \mathbf{R}^{1,1}\times\Sigma\times T^2$ where Σ is a 2-cycle in $K3$. We choose a vielbein frame such that, at least locally,

$$\hat e_M^{\ \underline{N}} = \begin{pmatrix} e_\mu^{\ \underline{\nu}} & 0 \\ e_A^{\ \underline{\nu}} & e_A^{\ \underline{B}} \end{pmatrix}. \tag{2.3}$$

Therefore, in the static gauge that we are considering, the induced metric on the M5-brane is simply $g_{\mu\nu} = \hat{g}_{\mu\nu}(X^{\underline{A}} = 0)$. We may further choose $\hat{\omega}_\mu{}^{\underline{\nu B}}(X^{\underline{A}} = 0) = 0$ and $\hat{\omega}_\mu{}^{\underline{\nu\lambda}}(X^{\underline{A}} = 0) = \omega_\mu{}^{\underline{\nu\lambda}}$, where $\omega_\mu{}^{\underline{\nu\lambda}}$ is the spin connection that one would calculate from the vielbein $e_\mu{}^{\underline{\nu}}$. Finally we also see that $\hat{\Gamma}_\mu = \hat{e}_\mu{}^{\underline{\nu}}\Gamma_{\underline{\nu}} = \Gamma_\mu$ is the same γ-matrix that one would calculate simply using the worldvolume metric $g_{\mu\nu}$.

This allows us reinterpret the bulk Killing spinor condition on the worldvolume as

$$\begin{aligned} 0 &= \hat{D}_\mu \epsilon \\ &= \partial_\mu \epsilon + \frac{1}{4}\hat{\omega}_\mu{}^{\underline{\nu\lambda}}\Gamma_{\underline{\nu\lambda}} + \frac{1}{4}\hat{\omega}_\mu{}^{\underline{AB}}\Gamma_{\underline{AB}} \\ &= D_\mu\epsilon + A_\mu\epsilon, \end{aligned} \tag{2.4}$$

where $\epsilon = \hat{\epsilon}(X^{\underline{A}} = 0)$, $\omega_\mu{}^{\underline{AB}} = \hat{\omega}_\mu{}^{\underline{AB}}(X^{\underline{A}} = 0)$ and $A_\mu = \frac{1}{4}\omega_\mu{}^{\underline{AB}}\Gamma_{\underline{AB}}$.

We find that, at lowest order in the fields X^A, $B_{\mu\nu}$ and ψ, the following symmetries close on-shell into translations, gauge transformations and local tangent frame rotations

$$\begin{aligned} \delta X^{\underline{A}} &= i\bar{\epsilon}\Gamma^{\underline{A}}\psi \\ \delta B_{\mu\nu} &= i\bar{\epsilon}\Gamma_{\mu\nu}\psi \\ \delta\psi &= \nabla_\mu X^{\underline{A}}\Gamma^\mu\Gamma_{\underline{A}}\epsilon + \frac{1}{2\cdot 3!}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\epsilon, \end{aligned} \tag{2.5}$$

where $\nabla_\mu X^{\underline{A}} = \partial_\mu X^{\underline{A}} + \omega_{\mu\underline{B}}{}^{\underline{A}} X^{\underline{B}}$. The Fermion equation of motion that is required to close the algebra is

$$\Gamma^\mu\nabla_\mu\psi = 0, \tag{2.6}$$

where $\nabla_\mu\psi = D_\mu\psi + A_\mu\psi$.

What are the remaining equations of motion? The B-field has a self-dual field strength $H = dB$ and hence one finds $d \star H = 0$. This condition is preserved by the supersymmetries (2.5). Taking a supersymmetry variation of the Fermion equation of motion (2.6) leads the condition

$$0 = \Gamma_{\underline{A}}\nabla^2 X^{\underline{A}}\epsilon + \frac{1}{2}\Gamma_{\underline{A}}\Gamma^{\mu\nu}F_{\mu\nu\underline{B}}{}^{\underline{A}}X^{\underline{B}}\epsilon, \tag{2.7}$$

where

$$\begin{aligned} F_{\mu\nu\underline{B}}{}^{\underline{A}} &= \partial_\mu\omega_{\nu\underline{B}}{}^{\underline{A}} - \partial_\nu\omega_{\mu\underline{B}}{}^{\underline{A}} + \omega_{\mu\underline{B}}{}^{\underline{C}}\omega_{\mu\underline{C}}{}^{\underline{A}} - \omega_{\mu\underline{B}}{}^{\underline{C}}\omega_{\mu\underline{C}}{}^{\underline{A}} \\ &= \hat{R}_{\mu\nu\underline{B}}{}^{\underline{A}}(X^{\underline{A}} = 0). \end{aligned} \tag{2.8}$$

To proceed we assume there is a relation of the form

$$\frac{1}{2}\Gamma_{\underline{A}}\Gamma^{\mu\nu}F_{\mu\nu\underline{B}}{}^{\underline{A}}\epsilon = M^{\underline{A}}_{\underline{B}}\Gamma_{\underline{A}}\epsilon, \tag{2.9}$$

in which case the equation of motion for $X^{\underline{A}}$, along with the other fields, is

$$\begin{aligned} \nabla^2 X^{\underline{A}} + M^{\underline{A}}_{\underline{B}}X^{\underline{B}} &= 0 \\ i\Gamma^\mu\nabla_\mu\psi &= 0 \\ H_{\mu\nu\lambda} &= \frac{1}{3!}\epsilon_{\mu\nu\lambda\rho\sigma\tau}H^{\rho\sigma\tau}. \end{aligned} \tag{2.10}$$

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We need to confirm that these equations are supersymmetric. To this end we note that the Fermion equation of motion (2.6) implies

$$\nabla^2\psi - \frac{1}{4}R\psi + \frac{1}{8}\Gamma^{\mu\nu}F_{\mu\nu}{}^{\underline{CD}}\Gamma_{\underline{CD}}\psi = 0. \tag{2.11}$$

To connect with (2.7) we multiply this on the left by $\bar{\epsilon}\Gamma^{\underline{A}}$ to find

$$\bar{\epsilon}\Gamma^{\underline{A}}\nabla^2\psi - \frac{1}{4}R\bar{\epsilon}\Gamma^{\underline{A}}\psi + \frac{1}{8}\bar{\epsilon}\Gamma^{\underline{A}}\Gamma^{\mu\nu}F_{\mu\nu}{}^{\underline{CD}}\Gamma_{\underline{CD}}\psi = 0. \tag{2.12}$$

Next we note that since $\hat{\omega}_\mu{}^{\underline{\nu A}} = 0$ we have that $\hat{R}_{\mu\nu}{}^{\underline{\lambda A}} = 0$ and therefore the Killing spinor integrability condition $[\hat{D}_\mu, \hat{D}_\nu]\epsilon = \frac{1}{4}\hat{R}_{\mu\nu}{}^{\underline{MN}}\Gamma_{\underline{MN}}\epsilon = 0$ implies

$$R_{\mu\nu}{}^{\underline{\lambda\rho}}\Gamma_{\underline{\lambda\rho}}\epsilon = -F_{\mu\nu}{}^{\underline{CD}}\Gamma_{\underline{CD}}\epsilon, \tag{2.13}$$

and hence

$$\bar{\epsilon}R = \frac{1}{2}\bar{\epsilon}\Gamma_{\underline{CD}}\Gamma^{\mu\nu}F_{\mu\nu}{}^{\underline{CD}}. \tag{2.14}$$

Using this we see that (2.12) implies

$$\bar{\epsilon}\Gamma^{\underline{A}}\nabla^2\psi + \frac{1}{2}\bar{\epsilon}\Gamma^{\underline{B}}\Gamma^{\mu\nu}F_{\mu\nu\underline{B}}{}^{\underline{A}}X^{\underline{B}}\psi = 0. \tag{2.15}$$

The $X^{\underline{A}}$ equation can be compared to (2.15) by noting that

$$\delta\nabla^2 X^{\underline{A}} = i\bar{\epsilon}\Gamma^{\underline{A}}\nabla^2\psi, \tag{2.16}$$

and one sees that the equations (2.10) are preserved by supersymmetry.

Let us consider for example the case where $\mathcal{W}$ is non-trivially embedded in eight dimensions, so that only $F_{\mu\nu}{}^{\underline{67}} \neq 0$. We see from (2.9) and (2.14) that the only non-vanishing components of $M^{\underline{A}}_{\underline{B}}$ are

$$M^{\underline{6}}_{\underline{6}} = M^{\underline{7}}_{\underline{7}} = R. \tag{2.17}$$

Thus we find the scalar equations are

$$\begin{aligned} \nabla^2 X^{\underline{6}} + RX^{\underline{6}} &= 0\,, \\ \nabla^2 X^{\underline{7}} + RX^{\underline{7}} &= 0\,, \\ \nabla^2 X^{\underline{A}} &= 0\,, \qquad A = 8, 9, 10. \end{aligned} \tag{2.18}$$

3. Counting moduli

In the previous section we determined the lowest order equation of motion for an M5-brane wrapped on a general calibrated submanifold $\mathcal{W}$ of $\mathcal{M}$. As a result we saw that the Fermions and scalar fields couple minimally to the gauge field associated to the structure group of the normal bundle and some scalars develop a mass term from the curvature. However the three-form remains closed and self-dual (at the linearized level). In this section we wish to

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perform a precise counting of the massless degrees of freedom for an M5-brane wrapped on a four-cycle $P \subset \mathcal{M}$, i.e. $\mathcal{W} = \mathbf{R}^{1,1} \times P$, in a spacetime of the form $\mathcal{M} = \mathbf{R}^{1,4} \times \mathcal{K}$ where $\mathcal{K}$ is some compact Calabi-Yau space that contains P. This has been discussed in great detail in [2, 9] and we will largely follow their discussion.

The simplest field to consider is the dimensional reduction of the two-form gauge field. As a consequence of the self-duality condition one finds $b_2^+(P)$ right moving scalars and $b_2^-(P)$ left moving scalars. For the compact Kähler manifolds that we consider here $b_2^+(P) = 2h_{2,0}(P) + 1$ and $b_2^-(P) = h_{1,1}(P) - 1$. If $h_{1,0}(P)$ is non-vanishing then there will be $2h_{1,0}(P)$ Abelian gauge fields in the two-dimensional effective theory. However these are non-dynamical we will not need them here.

Next we consider reduction of the scalars $X^{\underline{A}}$. In total there are five. Three of these, X^8, X^9, X^{10} simply parameterize the location in the non-compact transverse space. These always give 3 left and 3 right moving scalars in two dimensions. The remaining two scalars are in fact sections of the normal bundle of P inside $\mathcal{K}$. As such the number of such zero modes is hard to calculate. Let us denote the number of normal bundle moduli by $N(P, \mathcal{K})$. These are left-right symmetric and we will discuss them in more detail shortly.

As for the Fermions it is well known (see [11]) that spinors on a Kähler manifold P can be realized as $(0,p)$-forms on P. To see this one first consider complex coordinates for P so that $\{\Gamma^a, \Gamma^b\} = \{\Gamma^{\bar{a}}, \Gamma^{\bar{b}}\} = 0$ and $\{\Gamma^a, \Gamma^{\bar{b}}\} = 2g^{a\bar{b}}$ with $a, b = z, w$. In particular we consider a spinor ground state $|0\rangle$ which is annihilated by the holomorphic γ-matrices; $\Gamma^a|0\rangle = 0$. We can then construct a general spinor by

$$|\psi >= \omega|0\rangle + \Gamma^{\bar{a}}\omega_{\bar{a}}|0\rangle + \frac{1}{2}\Gamma^{\bar{a}\bar{b}}\omega_{\bar{a}\bar{b}}|0\rangle. \tag{3.1}$$

By construction $\omega_{\bar{a}_1\ldots\bar{a}_p}$ is totally anti-symmetric and hence represents a $(0,p)$-form on P. Furthermore if we choose the complex γ-matrices $\Gamma^z = \Gamma^2 + i\Gamma^4$ and $\Gamma^w = \Gamma^3 + i\Gamma^5$ then one sees that $\Gamma_{2345}|0\rangle = |0\rangle$ and more generally

$$\Gamma_{2345}|\omega_p\rangle = (-1)^p|\omega_p\rangle, \tag{3.2}$$

where $|\omega_p\rangle = \frac{1}{p!}\omega_{\bar{a}_1\ldots\bar{a}_p}\Gamma^{\bar{a}_1\ldots\bar{a}_p}|0\rangle$. Since the Fermions on the M5-brane satisfy $\Gamma_{012345}\psi = -\psi$ we see that $|\omega_p\rangle$ leads to right and left moving Fermions in two dimensions if p is even or odd respectively.

To find massless two-dimensional modes we assume that $|0\rangle$ is Killing with respect to ∇ defined above. In this case one see that solutions to $(\Gamma^a\nabla_a + \Gamma^{\bar{a}}\nabla_{\bar{a}})\psi = 0$ correspond to $\bar{\partial}_{[\bar{b}}\omega_{\bar{a}_1\ldots\bar{a}_p]} = 0$ and $g^{b\bar{a}_1}\partial_{[b}\omega_{\bar{a}_1\ldots\bar{a}_p]} = 0$, i.e. $\omega_p \in H^{(0,p)}(P)$. Thus one finds that number of massless left and right moving two-dimensional Fermions is

$$N_F^L = 4h_{1,0}(P), \qquad N_F^R = 4(h_{0,0}(P) + h_{2,0}(P)). \tag{3.3}$$

Here the factor of 4 comes from the fact the spinor 'groundstate' $|0\rangle$ can be thought of as having 32 real components but is subject to the three constraints: $\Gamma^z|0\rangle = \Gamma^w|0\rangle = 0$ and $\Gamma_{012345}|0\rangle = -|0\rangle$. Thus $|0\rangle$ has four real independent components.

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Let us summarize our counting so far. We find

$$\begin{aligned} N_B^L &= 2 + h_{1,1}(P) + N(P, \mathcal{K}) \\ N_B^R &= 4 + 2h_{2,0}(P) + N(P, \mathcal{K}) \\ N_F^L &= 4h_{1,0}(P) \\ N_F^R &= 4h_{2,0}(P) + 4, \end{aligned} \tag{3.4}$$

where we have assumed that $h_{0,0}(P) = 1$. Since the wrapped M5-brane preserves (at least) $(0, 4)$ supersymmetry the right-movers must have Bose-Fermi degeneracy. This immediately allows us to determine the number of normal moduli to be

$$N(P, \mathcal{K}) = 2h_{2,0}(P), \tag{3.5}$$

and hence the massless spectrum is

$$\begin{aligned} N_B^L &= 2h_{2,0}(P) + h_{1,1}(P) + 2 \\ N_B^R &= 4h_{2,0}(P) + 4 \\ N_F^L &= 4h_{1,0}(P) \\ N_F^R &= 4h_{2,0}(P) + 4. \end{aligned} \tag{3.6}$$

Note that this also ensures that the number of right-moving modes is a multiple of 4, as also required by $(0, 4)$ supersymmetry. We would like to emphasis that this formula should apply whenever it make sense to talk of a classical M-brane that is wrapped on a smooth complex submanifold of any smooth Calabi-Yau (including $K3 \times T^2$ and T^6).

This formula should be contrasted with the result

$$N(P, \mathcal{K}) = 2h_{2,0}(P) - 2h_{1,0}(P), \tag{3.7}$$

first obtained in [2] for an ample four-cycle P in a generic Calabi-Yau and extended to $K3 \times T^3$ and T^6 in [9]. We see that there is agreement for a generic Calabi-Yau where $h_{1,0}(P) = 0$. However, as pointed out in [9], the formula (3.7) contradicts supersymmetry when $h_{1,0}(P) \neq 0$. In the rest of this section we will argue that (3.5) is the correct counting and identify the missing modes that are absent from (3.7).

We start with a brief review of the calculation in [2]. This starts from the observation that a 4-cycle $P \subset \mathcal{K}$ is defined by the zeros of a section of a line bundle over $\mathcal{K}$. The Poincaré dual two-form to P, which we denote by $[P]$, determines the Chern class of the line bundle. Thus counting the number of deformations of P corresponds to counting the (real) dimension of the dimension of the space of line bundles. However one must take into account the fact that if P is described by zeros of a section s then the zeros of λs describe the same P for any $\lambda \in \mathbf{C}^\star$. Thus one needs the real dimension of the projective space of line bundles. In this way one determines $N(P,\mathcal{K})$ through

$$\begin{aligned} N(P,\mathcal{K}) &= 2\mathrm{dim}(H^0(\mathcal{K},\mathcal{L}))-2 \\ &= 2\sum_i(-1)^i\mathrm{dim}(H^i(P,\mathcal{L}))-2 \\ &= 2\int_{\mathcal{K}} e^{[P]}\mathrm{Td}(\mathcal{K})-2 \\ &= \frac{1}{3}\int_{\mathcal{K}}[P]^3+\frac{1}{6}\int_{\mathcal{K}}[P]\wedge c_2(\mathcal{K})-2. \end{aligned} \tag{3.8}$$

Here the second line follows from the Kodaira vanishing theorem; $H^i(\mathcal{K},\mathcal{L})=\emptyset$ for $i>0$, and the third line from a Riemann-Roch index formula. For an account of these theorems see [12, 13]. Next one can use the formula (see [2, 9])

$$h_{2,0}(P)=\frac{1}{6}\int_{\mathcal{K}}[P]^3+\frac{1}{12}\int_{\mathcal{K}}[P]\wedge c_2(\mathcal{K})+h_{1,0}(P)-1, \tag{3.9}$$

to obtain (3.7).

So what is missing from this calculation (3.8)? A central assumption of [2] is that P is an ample cycle. Technically this means that the Poincaré dual two-form $[P]$ lies inside the Kähler cone, i.e. it defines a positive volume for all complex 2-,4- and 6-cycles in $\mathcal{K}$. More intuitively an ample cycle P of a manifold $\mathcal{K}$ is one that is sufficiently generic so that the set of all normal vectors to P spans the entire tangent space of $\mathcal{K}$.

A key assumption of the Kodaira vanishing theorem is that the line bundle $\mathcal{L}$ is positive and hence (3.8) counts the dimension of the space of positive line bundles. While every ample four-cycle in $\mathcal{K}$ defines a positive line bundle there are zero modes which do not correspond to positive line bundles. In particular consider translations of P along any of the S^1 factors in $\mathcal{K}$. These S^1 factors are trivial and describing the location of an M5-brane in S^1 simply corresponds to specifying a value of the coordinate for that S^1. As such the location is simply a section of a trivial $\mathrm{U}(1)$ line bundle over P and this extends to a trivial $\mathrm{U}(1)$ bundle over $\mathcal{K}$. These deformations are not counted in (3.7) since the associated line bundle is trivial. There are $2h_{1,0}(\mathcal{K})$ such translations and, using the Lefschetz hyperplane theorem (valid for ample four-cycles), we have that $h_{1,0}(\mathcal{K})=h_{1,0}(P)$. Therefore we find an extra $2h_{1,0}(P)$ normal modes that arise from translations along the S^1 factors of $\mathcal{K}$. Including these modes in (3.8) gives (3.5).

An alternative description of these translational modes is to note that the S^1 factors are orbits of a $\mathrm{U}(1)$ Killing isometry that acts on $\mathcal{K}$. An ample cycle breaks the symmetries corresponding to translations along the S^1 factors and hence there must be $2h_{1,0}(\mathcal{K})=2h_{1,0}(P)$ Goldstone modes. There are also smooth but non-ample four-cycles for which $h_{1,0}(P)\neq h_{1,0}(\mathcal{K})$ and the index theorem does not apply. In these cases one also finds that the cycle breaks fewer $\mathrm{U}(1)$ isometries and as a result has fewer Goldstone modes. We will explicitly see in the examples below that nevertheless (3.5) is valid for all smooth four-cycles, as required by supersymmetry.

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3.1 Three examples

To illustrate this discussion let us consider some explicit examples for $\mathcal{K} = K3 \times T^2$. We will consider three choices for P: $P = K3$, $P = \Sigma \times T^2$ and $P = K3 + \Sigma \times T^2$, where Σ is a two-cycle in $K3$. For a useful account of various facts about $K3$ see [14]. Following this we will also discuss the case where $\mathcal{K} = T^6$.

First we consider the case where $P = K3$ which was first studied in detail in [8]. The Hodge diamond of $K3$ is

$$K3 : \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1\,. \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{3.10}$$

In this case it is clear that $N(K3, K3 \times T^2) = 2$ since $\mathcal{K}$ is simply a direct product $\mathcal{K} = K3 \times T^2$. Hence the normal bundle to $P = K3$ is trivial and there is no obstruction to moving the $K3$ around inside $\mathcal{K}$. Since the Killing spinor on $K3$ is chiral, reduction on $K3 \times T^2$ leads to a two-dimensional theory with $(0, 8)$ supersymmetry. Looking at the field content we find the massless modes

$$N_B^L = 24, \qquad N_B^R = 8\,, \qquad N_F^L = 0, \qquad N_F^R = 8, \tag{3.11}$$

which is the same as the worldsheet action for the Heterotic string on T^3.

Next we consider the case where $P = \Sigma \times T^2$. Let us suppose that Σ is ample in $K3$. In complex dimension two the Lefschetz hyperplane theorem does not imply that $h_{1,0}(\Sigma) = h_{1,0}(K3) = 0$ and hence $h_{1,0}(\Sigma) = g$ need not be zero. Assuming Σ is connected the Hodge diamond of $\Sigma \times T^2$ is

$$\Sigma \times T^2 : \qquad \begin{array}{ccccc} & & 1 & & \\ & 1+g & & 1+g & \\ g & & 2+2g & & g\,. \\ & 1+g & & 1+g & \\ & & 1 & & \end{array} \tag{3.12}$$

To determine $N(\Sigma \times T^2, K3 \times T^2)$ we note that $N(\Sigma \times T^2, K3 \times T^2) = N(\Sigma, K3)$. Since $K3$ does not have any S^1 factors we may use a similar calculation as in (3.8), suitably adapted to 2 complex dimensions. We find that

$$\begin{aligned} \dim(H^0(K3, \mathcal{L})) &= \sum_{i=0}^{2} (-1)^i \dim(H^i(K3, \mathcal{L})) \\ &= \int_{K3} e^{[\Sigma]} \mathrm{Td}(K3) \\ &= \int_{K3} \frac{1}{12} c_2(K3) + \frac{1}{2} [\Sigma]^2 \\ &= 2 + \frac{1}{2} \int_{K3} [\Sigma]^2. \end{aligned} \tag{3.13}$$

Thus proceeding as before and taking account of the projective equivalence we find

$$N(\Sigma, K3) = 2(\dim(H^0(K3,\mathcal{L})) - 1) = 2 + \int_{K3} [\Sigma]^2. \tag{3.14}$$

Continuing we observe that $[\Sigma]$ is the Poincare dual to Σ and hence we find

$$\begin{aligned} \int_{K3} [\Sigma]^2 &= \int_{\Sigma} [\Sigma] \\ &= -\int_{\Sigma} c_1(\mathcal{L}) \\ &= 2g - 2, \end{aligned} \tag{3.15}$$

where in the second line we have used the adjunction formula to identify $[\Sigma] = -c_1(\mathcal{L})$ and the last line follows from the well known formula for the Euler number of a two-dimensional surface. Note that ample implies that $g \geq 2$. Thus

$$N(\Sigma \times T^2, K3 \times T^2) = N(\Sigma, K3) = 2g. \tag{3.16}$$

Putting this all together we see that the field content is

$$N_B^L = 4 + 4g \qquad N_B^R = 4 + 4g \qquad N_F^L = 4 + 4g \qquad N_F^R = 4 + 4g. \tag{3.17}$$

Note that the spectrum is non-chiral which is a consequence of the fact that in this case $(4, 4)$ supersymmetry is preserved.

The previous two cases are not generic and in particular the cycle P is not ample in $\mathcal{K}$. In these cases the formula (3.7) does not necessarily apply and indeed it doesn't always agree with our results. However the formula (3.5) is valid and agrees with our discussion. Our final case $P = K3 + \Sigma \times T^2$ is generic in that P is ample. Therefore the calculation (3.8) is valid. However since $h_{1,0}(P) \neq 0$ we will be able to test whether (3.7) or (3.5) reproduces the correct number of normal modes. To see what these formulae give us we need to compute $h_{2,0}(P)$. From (3.9) we find

$$h_{2,0}(P) = \frac{1}{6}\int_{K3\times T^2} [P]^3 + \frac{1}{12}\int_{K3\times T^2} [P] \wedge c_2(K3 \times T^2) + h_{1,0}(P) - 1. \tag{3.18}$$

Writing $[P] = dvol_{T^2} + [\Sigma]$, where $dvol_{T^2}$ is the unit volume form of T^2, we find

$$\begin{aligned} h_{2,0}(P) &= \frac{1}{2}\int_{K3} [\Sigma]^2 + \frac{1}{12}\int_{K3} c_2(K3) + h_{1,0}(P) - 1 \\ &= g + h_{1,0}(P), \end{aligned} \tag{3.19}$$

where we have used (3.15) and $\chi(K3) = 24$ in the second line. Thus since P is ample $h_{1,0}(P) = 1$ and (3.7) predicts $2g + 2$ normal modes and (3.7) only $2g$ normal modes. However one expects that there are always two normal modes which come from translations along T^2 and also that all of the normal modes which exist in the embedding of Σ in $K3$ should also exist here. This example therefore demonstrates that the formula (3.7) fails to include the translational modes whereas (3.5) correctly accounts for all zero-modes.

For completeness we note that [2, 9]

$$\begin{aligned}h_{1,1}(P) &= \frac{2}{3}\int_{K3\times T^2}[P]^3 + \frac{5}{6}\int_{K3\times T^2}[P]\wedge c_2(K3\times T^2) + 2h_{1,0}(P)\\ &= 4g + 16 + 2h_{1,0}(P).\end{aligned} \tag{3.20}$$

Thus we find, from (3.6),

$$N_B^L = 6g+22, \qquad N_B^R = 4g+8\,, \qquad N_F^L = 4, \qquad N_F^R = 4g+8. \tag{3.21}$$

Finally we can follow the above discussion and consider what happens to these three cases when $K3$ is replaced by T^4, i.e. $\mathcal{K} = T^6$. In the first case where $P = T^4$ the same arguments give $N(T^4, T^6) = 2$ and hence

$$N_B^L = 8, \qquad N_B^R = 8\,, \qquad N_F^L = 8, \qquad N_F^R = 8, \tag{3.22}$$

which of course is just a straightforward reduction of the M5-brane on T^4 and has $(8,8)$ supersymmetry.

In the second case all we need to do is replace $c_2(K3) = 24$ by $c_2(T^4) = 0$ in (3.13) and we now find that the index theorem gives $2\dim(H^0(\Sigma, T^4) - 2 = 2g - 4$ normal modes. However we claim that, in addition to the modes counted by the index theorem, to obtain the number of normal mode deformations of Σ inside T^4 we must also include 4 translational Goldstone modes and hence $N(\Sigma, T^4) = 2g$. Thus we find $N(\Sigma\times T^2, T^6) = N(\Sigma, T^4) = 2g$. From (3.12) we find the total spectrum is

$$N_B^L = 4g+4, \qquad N_B^R = 4g+4\,, \qquad N_F^L = 4g+4, \qquad N_F^R = 4g+4. \tag{3.23}$$

Just as in the $K3\times T^2$ case this spectrum is non-chiral as result of enhanced $(4,4)$ supersymmetry.

In the third case where $P = T^4 + \Sigma\times T^2$ the cycle is ample we have that $h_{1,0}(P) = 3$. The calculations (3.19) and (3.20) give (replacing $c_2(K3) = 24$ by $c_2(T^4) = 0$ and setting $h_{1,0}(P) = 3$)

$$h_{2,0}(P) = g+1 \qquad h_{1,1}(P) = 4g+2. \tag{3.24}$$

Here we see that (3.7) gives $N(P, T^6) = 2g - 4$ and (3.5) gives $N(P, T^6) = 2g + 2$. The difference is 6 and these are clearly the translational modes along T^6 which must exist for a generic cycle which breaks all the translational symmetries. In total we find

$$N_B^L = 6g+6, \qquad N_B^R = 4g+8\,, \qquad N_F^L = 12, \qquad N_F^R = 4g+8, \tag{3.25}$$

In all these cases one finds that $N(P, T^6) = 2h_{2,0}(P)$ as predicted by (3.5). Let us make a comment on the first two cases where the cycles are not ample. In these cases the four-cycles preserve some of the symmetries of the torus and hence the total number of translational Goldstone modes (equal to 2 or 4 respectively) is less than $2h_{1,0}(\mathcal{K}) = 6$. Nevertheless we still find that the total number of normal modes is $2h_{2,0}(P)$.

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4. Counting black holes

Following [2] we can obtain black hole solutions of four-dimensional extended supergravity by further compactifying the remaining spatial direction of the wrapped M5-brane on S^1. A static wrapped M5-brane will be a magnetic source for the four-dimensional Abelian gauge fields that arise from Kaluza-Klein reduction of the M-theory three-form. One may also consider electric charges by including M2-branes. According to Beckenstein and Hawking the entropy of a macroscopic black hole is given by one quarter if its horizon area. For the solutions at hand one finds [2]

$$S_{\rm BH} = 2\pi\sqrt{\frac{1}{6}|q|\chi(P)}, \tag{4.1}$$

where $\chi(P) = 2 - 4h_{1,0}(P) + 2h_{2,0}(P) + h_{1,1}(P)$ is the Euler number of P. Here q is the momentum carried by the M5-brane along S^1, shifted by a contribution that arises from the electric charges [2].

Following the work of [1] one obtains the microscopic black hole degeneracy by counting all the modes of the low energy M5-brane theory which preserve the supersymmetries of the vacuum. For an effective theory with $(0,4)$ supersymmetry this amounts to counting the number of left moving modes with the right movers in their vacuum. From Cardy's theorem this is determined by the left-moving central charge and we find

$$S = 2\pi\sqrt{\frac{1}{6}|q|c_L} = 2\pi\sqrt{\frac{1}{6}|q|(\chi(P) + 6h_{1,0}(P))}, \tag{4.2}$$

where we have used (3.6). For a generic Calabi-Yau, where $h_{1,0}(P) = 0$, we see that the entropy is precisely reproduced by a microscopic counting of the degrees of freedom of the M5-brane, including the electric charges which in the microscopic picture arise from shifts of the vacuum energy [2]. However for $\mathcal{K} = K3 \times T^2$ or $\mathcal{K} = T^6$ we find $h_{1,0}(P) \neq 0$ and the two entropy calculations do not agree, as was pointed out in [9].

There is a further discrepancy. It is possible to compute the left and right central charges of the superconformal $(0,4)$ fixed point of the M5-brane using gravitational and R-symmetry anomalies [15, 16]. These arguments give $c_L = h_{2,0}(P)+h_{1,1}(P)+2-4h_{1,0}(P)$ and $c_R = 6(h_{2,0}(P) - h_{1,0}(P) + 1)$. This correctly accounts for the black hole entropy but also differs from our counting by $6h_{1,0}(P)$ for both the left and right central charges.

We propose the following resolution. Our field content naturally splits into that of a 'pure' $(0,4)$ supersymmetric sector with

$$\begin{aligned}
N_B^L &= 2h_{2,0}(P) + h_{1,1}(P) + 2 - 4h_{1,0}(P)\\
N_B^R &= 4h_{2,0}(P) + 4 - 4h_{1,0}(P)\\
N_F^L &= 0\\
N_F^R &= 4h_{2,0}(P) + 4 - 4h_{1,0}(P),
\end{aligned} \tag{4.3}$$

and $h_{1,0}(P)$ $(4,4)$ multiplets with

$$N_B^L = 4 \qquad N_B^R = 4 \qquad N_F^L = 4 \qquad N_F^R = 4. \tag{4.4}$$

Note that we are not assuming that there is a left-moving supersymmetry which acts on the $(4,4)$ multiplets, we are just using them as a counting device. The correct black hole degeneracy and central charges are readily obtained if we only count the modes of the $(0,4)$ sector. Furthermore both the black hole entropy and anomaly arguments only count the degrees of freedom that are massless at the conformal fixed point. Since the extra states that we find fall into non-chiral $(4,4)$ multiplets it is reasonable to conjecture that they become massive and hence do not appear in spectrum of the conformal fixed.

We have not been able to provide any additional arguments to support this proposal. However this claim is essentially a consequence of our counting along with the results of [15, 16]. To state this another way we note that the quantum anomaly arguments determine the central charges at the conformal fixed point and, combining this with our counting (which is just a classical counting at lowest order), we deduce that $h_{1,0}(P)$ $(4,4)$ multiplets become massive at the IR fixed point.

We note that $(4,4)$ supersymmetry implies that the potential must arise as the length-squared of a tri-holomorphic Killing vector on the moduli space [17]. When $h_{1,0}(P) \neq 0$ $\mathcal{K}$ has $\mathrm{U}(1)$ isometries and these will induce Killing vectors on the moduli space. We expect that, as a consequence of the geometrical action of R-symmetry, the moduli space Killing vectors should be tri-holomorphic. Therefore they can in principle lead to the required potential.

Let us make some comments on the mechanism that would provide such a mass. One could object that the M5-brane moduli cannot become massive because the equations of motion only involve derivatives of the fields. In particular, although in section 2 we only gave the lowest order equations of motion, the full non-linear equations have been worked out for a general embedding into supergravity and indeed these only involve derivatives of the fields. These equations of motion were derived in [10] using the superembedding formalism applied to the two derivative approximation to M-theory, i.e. standard eleven-dimensional supergravity of [18]. Another approach to obtaining the M5-brane equations of motion comes from an analysis of the Goldstone modes of the supergravity solution [19]. The M5-brane three-form H is identified with zero-modes arising from gauge transformations of the bulk three-form C. Again one would expect that, as Goldstone modes, the equations of motion of the M5-brane fields would only involve derivatives, even if higher derivative terms were added to eleven-dimensional supergravity.

However there is an important caveat. It is well-known that at next-to-leading order the M-theory effective Lagrangian contains the anomaly $C \wedge I_8$ term [20]. This leads to a source for C and hence the also three-form H on the M5-brane worldvolume. Furthermore it is precisely the $C \wedge I_8$ term in the effective action which is needed for cancelation of the anomalies and which ultimately leads to the correct prediction of the central charges in [15, 16]. Thus one might suspect that this term induces a mass for the extra $(4,4)$ multiplets that we have found.

We finish this section by noting that, in the examples above with $P = \Sigma \times T^2$ and $(4,4)$ supersymmetry, this mechanism removes all the massless modes. Thus the M5-brane will behave as though it is wrapped on a rigid cycle even though the cycle has moduli.

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5. Conclusion

In this paper we have discussed in detail the low energy dynamics of an M5-brane wrapped on a smooth but otherwise arbitrary, complex four-cycle in $K3 \times T^2$ or T^6. In particular we gave the lowest order equations of motion and determined the spectrum of massless modes. This required a careful treatment of the zero-modes that arise from translations along the S^1 factors and leads to a spectrum in a agreement with supersymmetry. Finally we discussed the counting of black hole microstates obtained by further reduction to four-dimensions on another S^1. The naive counting of massless modes does not reproduce the correct entropy and is not in agreement with anomaly cancelation arguments. To resolve this we proposed that $h_{1,0}(P)$ $(4,4)$ multiplets become massive and are removed from the low energy spectrum. It would be very interesting to study this mass mechanism in greater detail and verify that it indeed at work here. In particular it would be interesting to incorporate the effect of the $C \wedge I_8$ term on the M5-brane equations of motion.

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